U.G. 4th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : BMTMCCHT403

Course Title : Real Analysis-III

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) Show that $f(x)=\begin{cases} x, & 0\le x<1 \\ 1, & x=1 \end{cases}$ is Riemann integrable on [0, 1].

b) If $f(x)=\begin{cases} n; & \text{for } x=\frac{1}{n},\ n\in\mathbb{N}, \\ 0; & \text{otherwise} \end{cases}$ examine whether f is Riemann integrable or not on $[0,1]$.

c) Give examples of two non-Riemann integrable functions $f, g:[a,b]\to\mathbb{R}$ such that $fg\in\mathcal{R}[a,b]$.

d) Write down the Bessel's inequality with Fourier co-efficients a_n and b_n.

e) Consider the partition $P=\left(0, \frac{1}{2}, 1\right)$ of [0, 1]. Compute L(f, p) for $f(x)=-x^2,\ \forall x\in[0,1]$.

f) Give an example of a sequence of real valued continuous functions whose pointwise limit function is not continuous on a subset D of $\mathbb{R}$.

g) If R_1 and R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively, then what will be the radius of convergence of the power series $\sum_{n=0}^{\infty}(a_n+b_n)x^n$?

h) Show that, $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)=\frac{\pi}{2}$.

i) Does the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ converge uniformly for all real x?

j) Find the value of $B\left[\frac{1}{2}, \frac{1}{2}\right]$.

k) Define uniformly bounded sequence.

l) Give an example to establish if $f:[a, b] \to \mathbb{R}$ be Riemann integrable on [a, b] and $f(x) > 0$ for all $x \in [a, b]$, then $\frac{1}{f}$ may not be Riemann integrable on [a, b].

m) Find the limit of the sequence of partial sums of the series $\sum_{n=1}^{\infty}(1-x)x^n$.

n) Examine whether $\sum_{n=1}^{\infty}\cos nx$ is a Fourier series of some bounded integrable function over $[-\pi, \pi]$.

o) State Parsevel's identity for a 2π periodic function f on $[-\pi, \pi]$.

2. Answer any **five** questions: $2\times5=10$

a) Give an example of a trigonometric series which is everywhere convergent but not a Fourier series.

b) Let ϕ and ψ be two antiderivatives of a function f on [a, b]. Show that there exists $c \in \mathbb{R}$ such that $\phi = \psi + c$ on [a, b].

c) Let $f:[0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$.

Show that f has no antiderivative on [0, 1].

d) Find the radius of convergence of the power series $x + \frac{2x^2}{1!} + \frac{9x^3}{2!} + \frac{64x^4}{3!} + \ldots$.

e) Find the limit function of $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}$; $x \in [0, 2]$. Also state the reason whether the sequence of functions converges uniformly on [0, 2].

f) Let $\{f_n\}$ be a sequence of differentiable functions on [a, b] and $\{f_n\}$ be uniformly convergent to f on [a, b]. If f differentiable on [a, b]? Justify.

g) Show that the function $f(x) = \sum_{n=1}^{\infty}\frac{\cos(3^n x)}{2^n}$, $\forall x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

h) If $\sum_{n=1}^{\infty}a_n$ be a convergent series of real numbers then prove that the series

$a_1+\frac{a_2}{2^x}+\frac{a_3}{3^x}+\ldots$ is uniformly convergent on $[o, \infty]$.

3. Answer any **two** questions: 5×2=10

a) If $\{f_n\}$ be a sequence of Riemann integrable functions on [a, b] such that $\{f_n\}$ converges uniformly to a function f on [a, b]. Then prove that f is Riemann integrable on [a, b] and the sequence $\left\{\int_a^b f_n\right\}$ converges to $\int_a^b f$.

b) A function f is defined on [0, 1] by f(0)=0, $f(x)=(-1)^{n+1}(n+1)$, for $\frac{1}{n+1}<x\le\frac{1}{n}$ $(n=1, 2, 3, \ldots)$. Examine convergence of the integrals $\int_0^1 |f(x)|dx$.

c) Let $R(>0)$ be the radius of convergence of the power series $a_0+a_1x+a_2x^2+\ldots$. Prove that the radius of convergence of the power series series $a_1+2a_1x+3a_2x^2+\ldots+(n+1)a_{n+1}x^n$ obtained by term-by term differentiation is also R.

4. Answer any **one** question: 10×1=10

a) i) Prove that a bounded function f defined on $[a, b]$ is Riemann integrable over $[a, b]$ iff given $\varepsilon>0$, $\exists\delta>0$ such that $U(P, f)-L(P, f)<\varepsilon$ for every partition P of [a, b] satisfying $\|P\|<\delta$.

ii) Prove that,

$$\int_0^1 \frac{x^2dx}{(1-x^4)^{\frac{1}{2}}}\times\int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}}=\frac{\pi}{4\sqrt{2}}.$$

iii) Prove that the series

$$\sum_{n=1}^{\infty}\frac{x}{\{(n-1)x+1\}(nx+1)}$$

is not uniformly convergent on [0, 1].

4+4+2

b) i) If f(x) is the sum of the series $e^{-x}+2e^{-2x}+3e^{-3x}+\ldots$, $x>0$, then show that f is continuous for all $x>0$. Evaluate $\int_{\log 2}^{\log 3} f(x)\,dx$.

ii) Evaluate $\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ with justification.

6+4

c) i) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each $n \in \mathbb{N}$, let $f_n(x) = f\left(x + \frac{1}{n}\right)$, $\forall x \in \mathbb{R}$. Prove that $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

ii) If $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$ is the Fourier series of the $f(x) = \{\pi - |x|\}^2$, $\forall x \in [-\pi, \pi]$, then prove that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{n^4}{90}$.

4+6

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