U.G. 3rd Semester Examination - 2021

MATHEMATICS

Course Code : BMTMCCHT301

Course Title : Real Analysis-II

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) Evaluate $\lim_{x\to 0} \log|x|$.

b) Find $\frac{dx}{dy}$, when

$f(x,y)=\log(x^2+y^2)+\tan^{-1}\left(\frac{y}{x}\right)=0$.

c) Find the point of discontinuity of $f(x)=\operatorname{cosec} x$.

d) If f(x, y)=1, then find $\iint_R dx\,dy$ where R is the ractangle bounded by the lines x=a, x=b, y=c and y=d.

[Turn Over]

e) Find the stationary points of the function $f(x,y)=x^3+y^2-3x-6y-1$.

f) Show that $z=f(x^2y)$, where f is differentiable, satisfies $x\left(\frac{\partial z}{\partial x}\right)=2y\left(\frac{\partial z}{\partial y}\right)$.

g) Find the gradiant of $f(x,y,z)=x^2y^2+xy^2-z^2$ at the point (3, 1, 1).

h) Define directional derivative.

i) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous on $\mathbb{R}$ but is not differentiable only at 1.

j) Let $D\subset\mathbb{R}$ and $f:D\to\mathbb{R}$, $g:D\to\mathbb{R}$ be continuous functions. Show that $h(x)=\min\{f(x),g(x)\},\ \forall\, x\in D$ is continuous on D.

k) Define Lipschitz function on a interval $I\subset\mathbb{R}$.

l) Is $\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$ exists? Justify.

m) Find $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}$.

n) Find z_x if $x^3+y^3+z^3+6xyz=1$.

o) Examine the equality of f_{xy} and f_{yx} where $f(x,y)=x^3y+e^{xy^2}$.

2. Answer any **five** questions: 2×5=10

a) Show that $\log(1+x)$ lies between $x-\frac{x^2}{2}$ and $x-\frac{x^2}{2(1+x)}$, $\forall\, x>0$.

b) Given an example of function f and g which are not continuous at a point $c\in R$ but the product fg is continuous at c.

c) Evaluate $\iint_E \frac{x^2}{y^2}\,dydx$, E is bounded by x=2, y=x, xy=1.

d) Check the uniform continuity of the function $f(x)=\frac{\sin x}{x}$ on $(0,\infty)$.

e) A function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases} x^2, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$$

show that f is differentiable at 0 and $f'(0)=0$.

f) Change the order of the integration $\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$.

g) The plane x=1 intersets the surface $z = x^2+y^2$ in a parabola. Using partial derivatives find the slope of the tangent to the parabola at the point (1, 2, 5).

h) Find the directional derivative of $f(x, y)=x^2+y^2$ at (2, 2) in the direction (1, 1).

3. Answer any **two** questions: 5×2=10

a) A function f is defined on (–1, 1) by

$$f(x)=x^{\alpha}\sin\frac{1}{x^{\beta}},\ x\neq 0$$
$$=0 \qquad ,\ x=0.$$

Prove that (i) $0<\beta<\alpha-1$, f' is continuous at 0; (ii) if $0<\alpha-1\leq\beta$, f' is continuous at 0. 5

b) i) Using the $\delta-\varepsilon$ approach, find $\lim_{(x,y)\to(0,0)}\left[y+x\cos\left(\frac{1}{y}\right)\right]=0$.

ii) The cylinder $x^2+z^2=1$ is cut by the planes y=0, z=0 and x=y. Find the volume of the region in the first octant. 2+3

c) Show that

$$f(x,y)=\begin{cases} \frac{x^3-y^3}{x^2+y^2}, \text{ where } (x,y)\neq(0,0) \\ 0, \quad \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at (0, 0), possesses partial derivatives at (0, 0) but is not differentiable at (0, 0). 5

4. Answer any **one** question: 10×1=10

a) i) Calculate ξ in Cauchy's mean value theorem for the pair of functions $f(x)=\sin x,\ g(x)=\cos x$ on $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$.

ii) $f:R\to R$ is defined by $f(x)=|x|+|x-1|+|x-2|,\ x\in R$. Find the derived function f' and specify the domain of f'.

iii) Show that the function

$$f(x,y)=\begin{cases}(x+y)\sin\left(\dfrac{1}{x+y}\right); & x+y\neq 0\\ 0 & ;\ x+y=0\end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0). 3+3+4

b) i) A function $f:[0,1]\to[0,1]$ is continuous on [0, 1]. Prove that there exist a point c in [0, 1] such that $f(c)=c$.

ii) Let I be an interval and a function $f:I\to R$ be differentiable on I. Then $f'(I)$ is an interval.

iii) Find the extreme values of $f(x, y, z)=2x+3y+z$ such that $x^2+y^2=5$ and $x+z=1$. 3+2+5

c) i) If p(x) is a polynomial of degree >1 and $K\in\mathbb{R}$, prove that between any two real roots of p(x)=0, there is a real root of $p'(x)+kp(x)=0$.

ii) Prove that between any two real roots of the equation $e^x\cos x+1=0$, there is at least one real root of the equation $e^x\sin x+1=0\cdot$

iii) Show that

$$\iint_R\sqrt{4a^2-x^2-y^2}\,dxdy=\frac{4}{9}(3\pi-4)a^3,$$

where R is the upper half of the circle $x^2+y^2-2ax=0$. 4+3+3

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