U.G. 5th Semester Examination - 2021

MATHEMATICS

Course Code: BMTMCCHT 501

Course Title: Algebra-III

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: $1\times10=10$

a) Consider the group $(\mathbb{Z},+)$. Let $f:\mathbb{Z}\to\mathbb{Z}$ be defined by $f(n)=n+1$. Is this mapping a group homomorphism?

b) Show that for a group homomorphism $f:G\to G_1$, $f(e)=e_1$ where e and e_1 are the identity elements of G and G_1 respectively.

c) Find the correct answer:

The number of group homomorphisms from the cyclic group $\mathbb{Z}_6$ to the cyclic group $\mathbb{Z}_9$ is

i) 6 ii) 9

iii) 3 iv) 1

d) Give an example of an infinite non-cyclic group.

e) "$\mathbb{Z}/29\mathbb{Z}$ is an integral domain". Is it true? Justify.

f) Write down the assoiciated martix of the real quadratic form in three variables

$$x^2-2y^2+5z^2-2xy+4yz-16xz.$$

g) Prove that the subgroup z(G), the centre of a group G is normal in G.

h) Which of the following is not a field?

i) $\mathbb{Q}$ ii) $\mathbb{R}$

iii) $\mathbb{Z}[i]$ iv) $\mathbb{C}$

i) Why the rings $\mathbb{Z}$ and $2\mathbb{Z}$ are not isomorphic?

j) When an eigenvalue of a square matrix of order n is called regular?

k) Find a maximal ideal in the ring $\mathbb{Z}_6$.

l) Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be given by $T(x, y)=(x, y, xy)$ for all $(x, y)\in\mathbb{R}^2$. Is T a linear transformation? Give reason.

m) Prove that $\mathbb{Z}_2\times\mathbb{Z}_2$ is not a cyclic group.

n) If $\varphi:G\to G^1$ be a group homomorphism with $\operatorname{Ker}\varphi=\{e\}$. Prove that φ is injective.

o) Prove or disprove : If (–1) is an eigen value of an n×n matrix A, then det $(A^2-I_n)=0$.

2. Answer any **five** questions: 2×5=10

a) Let G be a non-commutative group of order 10. Then show that G has a trival centre.

b) Show that the ideal $7\mathbb{Z}$ in the ring $\mathbb{Z}$ is a prime ideal.

c) Consider the ring C[a, b] of all real valued continuous functions defined on [a, b]. Show that for any $c \in [a,b]$, the set $\{f \in C[a,b] : f(c)=0\}$ is a maximal ideal of c[a, b].

d) Show that the matrix $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ is not diagonalisable.

e) Let $\langle V, <.,.> \rangle$ be an inner product space and $<x,y>=<x,z>$ for all $x \in V$. Then show that y = z.

f) Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. Such that T(1, –1, 1)=(1, 0) and T(1, 1, 1)=(0, 1)? Explain your answer.

g) In the ring $\mathbb{Z}$, prove tht the ideal $p\mathbb{Z}$ is a prime ideal if p is a prime number.

h) If G is an infinite cyclic group, prove that Aut(G) is a group of order 2.

3. Answer any **two** questions: 5×2=10

a) i) Let G be a group in which $(ab)^3 = a^3b^3$ for all $a,b \in G$. Then prove that $H = \{x^3 : x \in G\}$ is a normal subgroup of G.

ii) Is the ideal $I = \{f \in C[0,1] : f(0)=f(1)=0\}$ a maximal ideal in the ring C[0, 1]?

3+2=5

b) i) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x,y,z)=(3x, x-y, 2x+y+z)$. Is T invertible? If so, find a rule of T^{-1}.

ii) If x and y be two orthogonal vectors in a Euclidean space V then prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. 3+2=5

c) Suppose V be a finite dimensional vector space over a field F and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be an ordered basis of V. Let $T : V \to V$ be a linear operator such that

$T(\alpha_1)=\alpha_2,\ T(\alpha_2)=\alpha_3,\ldots,T(\alpha_{n-1})=\alpha_n,$
$T(\alpha_n)=\alpha_1.$

Prove that $T^n = I$, where I is the identity mapping of V.

If λ be an eigen value of an $n\times n$ idempotent matrix A, prove that λ is either 1 or 0.

4+1=5

4. Answer any **one** question: 10×1=10

a) i) If H is the only subgroup of order n in a group G then prove that H is a normal subgroup of G.

ii) State and prove second isomorphism theorem of groups. 4+(2+4)=10

b) i) Is there any integral domain which has six elements?

ii) Prove that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic.

iii) Suppose F is a field and there is a ring homomorphism from $\mathbb{Z}$ onto F. Show that $F \simeq \mathbb{Z}_p$ for some prime number p.

3+3+4=10

c) i) If G and G' be two group and $\varphi : G \to G'$ be an onto homomorphism, prove that the quotient group $G/\ker\varphi$ is isomorphic to G'.

ii) Find the permutation group isomorphic to Klein's 4-group.

iii) Prove that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z)=(x+y, y+z, z+x), (x, y, z)\in\mathbb{R}^3$ is one to one and onto.

5+2+3=10

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