U.G. 5th Semester Examination - 2021

MATHEMATICS

Course Code: BMTMCCHT 502

Course Title: Metric Spaces and Complex Analysis

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: $1\times10=10$

a) Find the interior of the set [a, b] a, b$\in\mathbb{R}$ with respect to $\mathbb{R}$ with discrete metric.

b) What is the smallest closed set containing $\mathbb{R}$–Q in $\mathbb{R}$ with respct to usual metric?

c) Give an example of a proper open dense subset of $\mathbb{R}$ with respect to usual metric.

d) Find he diameter in $\mathbb{N}$ with respect to $\mathbb{R}$ with discrete metric.

e) Let $\{F_n \; n\in\mathbb{N}\}$ be a nested sequence of nonempty sets in a metric space with diam

$(F_n)\to 0$ as $n\to\infty$. Is $\bigcap_{n=1}^{\infty} F_n \neq \phi$? Justify.

f) Does there exist an analytic function of $\mathbb{C}$ with real part $u(x,y)=y^2$? Justify.

g) Show that $f(z)=z, \forall z\in\mathbb{C}$ is conformal on $\mathbb{C}$.

h) Show that $\lim_{z\to0}\frac{\mathrm{Re}(z)}{z}$ does not exist.

i) If $A=\{(x,y):x^2+y^2=1\}$ and $B=\{(x,y):(x-2)^2+y^2=2\}$. Find diam $(A\cup B)$ with rspect to the usual metric on $\mathbb{R}^2$.

j) Let $S^1=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ be the unit circle in $\mathbb{R}^2$. Is S^1 closed? Explain.

k) When two metrices are said to be equivallent?

l) Prove that isometry is an one-to-one map.

m) Let a function f be differentiable at $z_0\in\mathbb{C}$. Prove that the function f is continuous at z_0.

n) Find the fixed points of the bilinear transformation $\omega=\frac{3z-4}{z-1}$.

o) Find the radius of convergence of the power series $\sum\left(\frac{n+r}{n}\right)^{n^2} z^n$.

2. Answer any **five** questions: 2×5=10

a) Show that $d(x,y)=\left|x^4-y^4\right|, \forall x,y \in \mathbb{R}$ is not a metric on $\mathbb{R}$.

b) Find the diameter of the set $\{(x,y): 0<x<1, y=e^x\}$ with respect to $\mathbb{R}^2$ with usual metric.

c) Draw the closed ball centre at (0, 0) radius 1 with respect to $\mathbb{R}^2$ with metric, $d((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|,|y_1-y_2|\}$, $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$.

d) Show that $f(z)=\mathrm{Re}(z), \forall z \in \mathbb{C}$ is nowhere differentiable but everywhere continuous on $\mathbb{C}$.

e) Show that Cantor's intersection theorem does not hold for the family of open intervals.

f) Three complex numbers z_1, z_2, z_3 are such that $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|$. Prove that they represent the vertices of and equilateral triangle.

g) Show that $w=iz+i$ maps half plane x>0 into half plane v >1.

h) Show that the function $f(z)=\bar{z}$ is non-analytic everywhere in $\mathbb{C}$.

3. Answer any **two** questions: 5×2=10

a) Let (Y,d') be a subspace of a metric space (X, d). Then prove that a set ACY is open in (Y,d') if and only if there exists an open set G in (X, d) such that A = G∩Y. 5

b) i) Prove that if a power series $\Sigma a_n z^n$ converges when $z=z_1(\neq 0)$, then it is absolutely convergent for every value of z such that $|z|<|z_1|$.

ii) Find the radius of convergence of power series

$$\sum_{n=1}^{\infty} a_n z^n \text{ where } a_n \begin{cases} 0, \text{ if } n \text{ is even} \\ n, \text{ if } n \text{ is odd} \end{cases}.$$

3+2=5

c) Prove that every separable metric space is second countable. 5

4. Answer any **one** question: 10×1=10

a) i) Show that a subset Y of a complete metric space (X, d) is complete if and only if Y is closed in (X, d).

ii) Show that $\mathbb{N}$ is complete with respect to usual metric on $\mathbb{R}$.

iii) Find a bilinear transformation which maps z = 0, –i, –1 into w = i, 1, 0 respectively. 4+2+4=10

b) i) Prove that continuous image of a connected set is connected.

ii) Express the relation $\omega = \frac{13iz+75}{3z+5i}$ in the form $\frac{\omega - a}{\omega - b} = k.\frac{z-a}{z-b}$ where a, b, k are constants. Show that the circle in the z-plane whose centre is z=0 and radious 6 transformed into th circle in the w-plane on the line joining w=a and w=b as diameter, and the points in the z -plane which are exterior to the former circle are transformed into the points in the w-plane within the latter circle.

5+(1+2+2)=10

c) i) Show that the diagonal $\{(x,x): x \in X\}$ is closed in the product metric space X×X.

ii) Let (X, d) be a metric space without any isolated point and Y be a dense subset of X. Show that for any non-empty open set U of X, $U \cap Y$ is infinite.

iii) Find the analytic function $w = u + iv$, where $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$.

3+3+4=10

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