U.G. 4th Semester Examination - 2021

MATHEMATICS

Course Code : BMTMCCHT 403

Course Title : Real Analysis-III

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) Let $f(x)=\frac{1}{x}$ on $(0,1]$ and P_n be the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ of $(0,1]$. Compute $L(P_n, f)$.

b) Give an example of a function which is integrable but primitive does not exist.

c) State 2nd MVT of integral calculus in Bonnet's form.

d) Show that, $\int_0^\infty \sqrt{x}.e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$.

e) Define Riemann sum of a bounded function on [a, b].

f) Evaluate: $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$.

g) Examine whether fundamental theorem of integral calculus is applicable to evaluate the integral $\int_0^3 x[x]\,dx$.

h) Using Abel's test show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}.x^n}{n^p(1+x^n)};\ 0 < p \le 1$$

is uniformly convergent on [0, 1].

i) Examine whether the sequence $\left\{\frac{x^n}{1+x^n}\right\}$; $0 \le x \le 2$ converges uniformly on [0, 2].

j) Evaluate: $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$

k) Find the radius of convergence of the power series $\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \ldots$.

l) If f(x) be non-negative continuous function on [a, b] and $\int_a^b f(x)\,dx = 0$, then show that $f(x) = 0;\ \forall x \in [a, b]$.

m) State the Darboux theorem on upper and lower sum for all partitions P of [a, b] satisfying $\|P\| \le \delta$.

n) Show that Riemann integrals satisfy linearity properties.

o) Define uniform convergence for a series of functions.

2. Answer any **five** questions: 2×5=10

a) Prove that, $2^n \Gamma\left(n + \frac{1}{2}\right) = 1.3.5...(2n-1)\sqrt{\pi}$.

b) Evaluate: $\lim_{x \to 0} \frac{x}{1 - e^{x^2}} . \int_0^x e^{t^2} dt$ with proper justification.

c) For a function f, continuous on [0, 1] show that $\lim_{n \to \infty} \int_0^1 \frac{nf(x)}{1 + n^2x^2} dx = \frac{\pi}{2} f(0)$.

d) Let $f : [0, 1] \to \mathbb{R}$ be a function defined by,

$$f(x) = 1 \; ; \quad x = 0$$
$$= x \; ; \quad 0 < x \le 1.$$

Show that $\frac{1}{f}$ is not integrable on [0, 1].

e) If $f(x)=x^2+x$ is expresses as a Fourier series in (–2, 2) to which values this series converges at x=2?

f) Examine the convergence of the sequence $\{(x_n, y_n)\}_n$ where $x_n = \left(1 + \frac{1}{n}\right)^n$ and $y_n = \sqrt[n]{n}$.

g) Prove that

$$1 + e^{-x} \cos x + e^{-2x} \cos 2x + ... + e^{-nx} \cos nx + ...$$

converges uniformly on a set $S(\subset \mathbb{R})$ which is bounded below by a positive constant.

h) Show that the second MVT of integral calculus in Weierstrass' form does not hold in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the function $f(x) = \cos x$ and $g(x) = x^2$.

3. Answer any **two** questions: 5×2=10

a) If Q be any refinement of a partition P of [a, b], satisfying $\|P\| \le \delta$, containing exactly k additional points of division than P. Then for any bounded function $f : [a, b] \to \mathbb{R}$ defined on [a, b] prove that,

$$0 \le U(P, f) - U(Q, f) \le (M - m).k.\delta$$

and $0 \le L(Q, f) - L(P, f) \le (M - m).k.\delta$

where, M and m are the supremum and infimum of f on [a, b] respectively.

b) Find the Fourier series of $f(x) = x \sin x$ in $[-\pi, \pi]$ and deduce that,

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots.$$

c) Let $f_n(x) = \frac{nx}{1+nx}$, $\forall x \in [0, 1]$, $\forall n \in \mathbb{N}$.

Show that the sequence of function $\{f_n\}$ converges to a function f on [0, 1] but the convergence is not uniform on [0, 1].

4. Answer any **one** question: 10×1=10

a) i) Let $f:[a, b] \to \mathbb{R}$ be a function defined and bounded on [a, b] such that f is continuous except for a finite number of points of [a, b]. Show that f is Riemann integrable.

ii) State and prove Abel's test for convergence of an improper integral.

5+5

b) i) Let f be a function Riemann integrable on [a, b]. Then,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a),$$

where M and m are the supremum and infimum of f on [a,b].

Consequently, $\int_a^b f(x)\,dx = \mu(b-a)$ for some μ with $m \le \mu \le M$.

Also if f is continuous on [a, b]. Then there exists a point ξ in [a, b] such that, $\int_a^b f(x)\,dx = (b-a)f(\xi)$.

ii) Assuming the convergence of the integral show that,

$$\int_0^1 \log\Gamma(x)\,dx = \frac{1}{2}\log(2\pi).$$ (3+2+1)+4

c) i) State and prove Cauchy Hadamard test for determination of radius of convergence of a power series.

ii) Find the radius of convergence of the power series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \ldots.$$ 8+2

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