U.G. 3rd Semester Examination - 2020

MATHEMATICS

Course Code : BMTMCCHT301

Course Title : Real Analysis-II

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) Pick out the true statement:

i) Every function from $\mathbb{N}$ to $\mathbb{R}$ is continuous.

ii) There exists a function from $\mathbb{N}$ to $\mathbb{R}$ which is not continuous.

b) If $f(x,y)=e^{\sin\left(\frac{x}{y}\right)}$, write down $f_{xy}(x,y)$ at $\left(\frac{\pi}{2},1\right)$.

c) Change the order of the integration $\int_{y=0}^{1}\int_{x=0}^{y+4}\frac{2y+1}{x+1}dxdy$.

d) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x)=0,\ \forall x\in Q$. Does $f(x)=0,\ \forall x\in\mathbb{R}$?

e) Let $D\subset\mathbb{R}$ and $f:D\to\mathbb{R}$, $g:D\to\mathbb{R}$ be continuous functions. Show that $h(x)=\max\{f(x),g(x)\},\ \forall x\in D$ is continuous on D.

f) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous only at 1.

g) Prove that $f(x)=\frac{1}{x^2},\ \forall x\in(0,1)$ is not uniformly continuous on (0, 1).

h) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is monotonically increasing on $\mathbb{R}$ but not continuous on $\mathbb{R}$.

i) Let I be an interval of $\mathbb{R}$ and $f:I\to\mathbb{R}$ be a function such that f has a minimum at $C\in I$. Does $f'(c)=0$?

j) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is not differentiable only at 1.

k) Let f be continuous of R. Then show that $A=\{x\in R \mid f(x)>0\}$ is an open set.

l) Give an example of a function which is continuous at a point but not derivable at that point.

m) Let $f(x,y)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x\le 0\end{cases}$.

Does $f_x(0,0)$ exist? Justify.

n) Obtain the fourth degree Taylor's polynomial approximation to $f(x)=e^{2x}$ about x=0.

o) Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function such that f is differentiable at $(a,b)\in\mathbb{R}^2$. Then which of the following is true

i) f has directional derivative at (a, b) in any direction.

ii) f does not have directional derivative at (a, b) in some direction.

2. Answer any **five** questions: 2×5=10

a) State Cauchy's mean value theorem.

b) Let $f:[0,2]\to\mathbb{R}$ be continuous and $f(0)=f(2)$. Prove that there exists a $c\in[0,1]$ such that $f(c)=f(c+1)$.

c) Let $f:[-1,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}1 & \text{if } x\in[-1,1]\cap Q\\ -1 & \text{if } x\in[-1,1]\cap(\mathbb{R}-Q).\end{cases}$$

Does there exist a function g such that $g'(x)=f(x), \forall x\in[-1,1]$?

d) If $c_0+\frac{c_1}{2}+\frac{c_2}{3}+\ldots+\frac{c_n}{n+1}=0$, where $c_0,c_1\ldots,c_n\in\mathbb{R}$, show that the equation $c_0+c_1x+c_2x^2+\ldots+c_nx^n=0$ has at least one real root between 0 and 1.

e) Let $f:[a,b]\to\mathbb{R}$ be such that $|f(x)-f(y)|\le M|x-y|^2, \quad \forall x,y\in[a,b]$ for some real number M>0,. Show that f is constant on [a, b].

f) If $\lim_{x\to G} f(x)=l$, then show that $\lim_{x\to G}|f(x)|=|l|$. Is its converse true?

g) Find the directional derivative of $f(x,y)=x^3-3xy+4y^2\ \forall(x,y)\in\mathbb{R}^2$ at (0, 0) in the direction of the line that makes an angle of $\frac{\pi}{6}$ with the x-axis.

h) Evaluate the double integral $\iint_R e^{x^2}dxdy$, where, the region R is given by $R: 2y\le x\le 2$ and $0\le y\le 1$.

3. Answer any **two** questions: 5×2=10

a) i) Let I be an interval. Prove that if $f:I\to\mathbb{R}$ be such that f' exists and bounded on I, then f is uniformly continuous on I.

ii) Show that the function $f(x)=\frac{1}{1+x^2}, \forall x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 3+2

b) Let I=[a,b] be a closed and bounded interval and $f:[a,b]\to\mathbb{R}$ be continuous on I. Then prove that $f(I)=\{f(x): x\in I\}$ is a closed and bounded interval. 5

c) i) Show that $f(x,y)=(y-x)^4+(x-2)^4, \forall (x,y)\in \mathbb{R}^2$ a minimum at (2, 2).

ii) Show that $\lim_{(x,y)\to(0,0)}\sqrt{1-x^2-y^2}=1$ using $\varepsilon-\delta$ definitions. 2+3

4. Answer any **one** question: 10×1=10

a) i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases}2x, & x\in Q\\ 1-x, & x\in\mathbb{R}-Q\end{cases}$ Prove that f is continuous at $\frac{1}{3}$ and discontinuous at every other point.

ii) A function $f:[0,1]\to\mathbb{R}$ is continuous on [0, 1] and f assumes only rational values on [0, 1]. Prove that f is constant on [0, 1].

iii) Let $c\in\mathbb{R}$ and a function $f:\mathbb{R}\to\mathbb{R}$ is continuous at c. If for every positive δ there is a point y in $(c-\delta, c+\delta)$ such that f(y)=0, prove that f(c)=0.

5+2+3

b) i) If $f(x+y)=f(x)+f(y)$, for all $x, y\in\mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$.

ii) Evaluate the integral of $f(x,y,z)=1$ over a tetrahedron with vertices at (0,0,0), (1,1,0), (0,1,0), (0,1,1). 4+6

c) i) A function f is defined on some neighbourhood of c and $\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$ exist. Does $f'(c)$ exist?

ii) Verify Rolle's Theorem for the function $f(x)=e^x \sin x$ in $[0,\pi]$.

iii) Find the points closest to the origin on the hyperbolic cylinder $x^2-z^2=1$.

2+3+5

———————