U.G. 5th Semester Examination - 2020

MATHEMATICS

Course Code: BMTMCCHT 502

Course Title: Metric Spaces and Complex Analysis

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 1×10=10

a) Give an example of a nowhere dense set.

b) Is every open ball in a metric space infinite set? Justify.

c) What is the smallest closed set containing Q in $\mathbb{R}$ with respect to usual metric?

d) What is the largest open set contained in [1, 2] with respect to $\mathbb{R}$ with discrete metric?

e) Prove that in a discrete metric space any set cannot have any limit point.

f) Write down the Cauchy-Riemann equations.

g) Show that union of two bounded sets in a metric space is bounded.

h) Is $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ convergent in $\mathbb{R}$ with discrete metric? Justify.

i) For every complex number z, show that $\left|e^z\right| \le e^{|z|}$.

j) Show that $f(z) = e^z,\ \forall z \in \mathbb{C}$ is conformal on $\mathbb{C}$.

k) Show that $f(z) = \mathrm{Re}(z),\ \forall z \in \mathbb{C}$ is nowhere analytic.

l) Find the stereographic projection of the point (2, 3, 0).

m) Show that $\lim\limits_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

n) Define Möbius transformation.

o) Show that harmonic conjugates of a harmonic function u differ by a constant in a region G of $\mathbb{C}$.

2. Answer any **five** questions: 2×5=10

a) Show that $d(x, y) = |e^x - e^y|$, $\forall\, x, y \in \mathbb{R}$ is a metric on $\mathbb{R}$.

b) Let 'A' be a subset of metric space (X, d). Show that if A is bounded in (X, d), then $\exists\, a \in X$, $r > 0$ real number such that $A \subset B_r(a)$.

c) Give examples to show that, if two sets are separated then their complements may or may not be separated.

d) Prove that in a discrete metric space (X, d), X is connected iff it is a singleton set.

e) Prove or disprove: Every subset of $\mathbb{N}$ is open in $\mathbb{N}$.

f) Show that $f(z) = |z|$, $\forall\, z \in \mathbb{C}$ is nowhere differentiable but everywhere continuous in $\mathbb{C}$.

g) Let G be a region in $\mathbb{C}$ and $f: G \to \mathbb{C}$ be an analytic function. Show that if f assumes only real values on G, then f is constant on G.

h) Find the radius of convergence of the power series, $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)z^{n^2}$.

3. Answer any **two** questions: 5×2=10

a) Let (X, d) be a metric space and Y be separable and dense in X. Show that X is separable. 5

b) i) Show that in a metric space, every open set is union of open balls.

ii) If the real part of the complex number $\frac{z-i}{z-1}$ is zero, then show that the complex number z lies on the circle with centre $\frac{1+i}{2}$ and radius $\frac{1}{\sqrt{2}}$. 3+2

c) i) Prove that a discrete metric space (X, d) is separable iff X is countable.

ii) Prove that if the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is R, then the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R. 2+3

4. Answer any **one** question: 10×1=10

a) i) Let (X, d) and (Y, d') be two metric spaces. Show that a function $f:(X, d) \to (Y, d')$ is continuous iff for all sets $A \subset X$,

$$f(\bar{A}) \subset \overline{f(A)}.$$

ii) Show that continuous image of a separable metric space is separable.

iii) let A be a non-empty subset of a metric space (X, d). Show that the function $f:(X, d) \to \mathbb{R}$ given by $f(x) = d(x, A)$, $\forall\, x \in X$ is continuous. (here consider usual metric on $\mathbb{R}$) 4+3+3

b) i) For a subset A of a metric space (X, d), prove that A is closed iff every sequence in A, which converges in X, converges to a point of A.

ii) Show that the convex combination of two metric is again a metric.

iii) If the mapping of z-plane upon w-plane be conformal, then show that the only form of transformation is w=f(z); where f(z) is an analytic function of z.

4+3+3

c) i) Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and find its conjugate harmonic.

ii) Let (X, d) and (Y, d') be two metric spaces. A function $f:(X, d) \to (Y, d')$ is continuous if and only if for all closed sets F in (Y, d'), $f^{-1}(F)$ is closed in (X, d). Prove this.

iii) Let (X, d) be a metric space and A is a non-empty subset of X. Prove that A is disconnected iff we can express $A \subset G_1 \cup G_2$, where G_1 and G_2 are non-empty open sets in (X, d) such that $A \cap G_1 \neq \phi$, $A \cap G_2 \neq \phi$ but $A \cap (G_1 \cap G_2) = 4$. 3+3+4

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